Recent Trends in the Study of Compact Objects (RETCO-II): Theory and Observation
ASI Conference Series, 2015, Vol. 12, pp 1 – 8
Edited by Indranil Chattopadhyay, Anuj Nandi, Santabrata Das and Samir Mandal

General Relativity and Compact Objects

Patrick Das Gupta*
Department of Physics and Astrophysics, University of Delhi, Delhi - 110 007 (India)

Abstract. Two of the characteristic predictions of general relativity (GR) - black holes (BHs) and gravitational waves (GWs), have played significant roles in relativistic astrophysics, e.g. compact X-ray sources, AGNs, GRBs, coalescing binary pulsars, etc. BHs also emit thermal Hawking radiation due to quantum processes in its vicinity. Effect of Hawking radiation on the fate of 'black hole atoms' has been discussed in this paper. In the context of GWs, discovery of RX J0648.0-4418, a rotating white dwarf (WD) in a binary system with mass $\approx 1.2\ M_\odot$ and spin period 13.2 s, motivates one to revisit the problem of spin evolution due to emission of gravitational as well as electromagnetic radiations from rapidly spinning magnetized WDs.

Keywords : gravitation – relativity – black hole physics – gravitational waves

1. Introduction

Einstein's general relativity (GR) is a relativistic theory of gravitation that rests on three pillars - (a) equivalence principle, (b) special relativity (SR) and (c) space-time geometry. Because of (a), no matter how strong the gravity is or how rapidly it varies with space-time, one can always choose a local inertial frame (LIF) of limited extent in space and time such that gravity disappears in it (although the gravitational tidal field does not). Such a LIF corresponds to a freely falling frame in which the metric tensor is simply the Minkowski metric $\eta_{\mu\nu}$ of SR. Armed with (a) and (b), GR insists that mathematical forms of all non-gravitational physical laws in the LIFs take the same corresponding forms as they would in 'gravity free' inertial frames of SR. One can discern from (a) a link between gravity and space-time geometry, as e.g. no

*email: patrick@srb.org.in

matter how curved a 2-dimensional surface is, one can always choose a sufficiently small patch Σ on it such that the distance between any two nearby points on Σ can be obtained from $dl^2 = dx^2 + dy^2$ of Euclidean geometry.

To connect LIFs at different space-time points, and to express physical laws in terms of arbitrary coordinates in reference frames of size as large as one wishes, one uses tensor calculus (or differential geometry) to acquire an affine connection $\Gamma^{\mu}_{\alpha\beta}(x^{\lambda})$ derivable from the metric tensor $g_{\mu\nu}(x^{\lambda})$ and its derivatives. Although, $\Gamma^{\mu}_{\alpha\beta}(x^{\lambda})$ vanishes at a point in a LIF, its derivative does not. Tidal force, which represents genuine gravity, is related to the Riemann curvature tensor $R^{\mu}_{\nu\alpha\beta}(x^{\lambda})$, a fourth rank tensor constructed out of the connection $\Gamma^{\mu}_{\alpha\beta}$ and its derivatives. $R^{\mu}_{\nu\alpha\beta}$ determines whether space-time is flat or curved. In the gauge theory framework, $\Gamma^{\mu}_{\alpha\beta}$ is analogous to gauge potential with $R^{\mu}_{\nu\alpha\beta}$ as the corresponding gauge covariant field strength.

The notion of gravitational mass becomes superfluous in GR since particle trajectories are geodesics of space-time geometry determined from the line-element,

$$ds^2 = g_{\mu\nu}\, dx^{\mu} dx^{\nu} \ . \tag{1}$$

Therefore, it is not surprising that the world lines of freely falling test particles are independent of their inertial masses. On the other hand, the dynamics of space-time geometry is determined by the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{2}$$

where, the Ricci tensor and Ricci scalar are $R_{\mu\nu} \equiv R^{\alpha}_{\mu\alpha\nu}$ and $R = g^{\mu\nu} R_{\mu\nu}$, respectively. In eq.(2), $T_{\mu\nu}$ is the matter energy-momentum tensor whose components represent the flux of energy and momentum carried by matter in appropriate directions. For a non-relativistic and static distribution of matter density ρ with negligible pressure, eq. (2) reduces, in the weak field limit, to Newtonian gravity $\nabla^2 \phi = 4\pi G \rho$, where the gravitational potential $\phi \approx (g_{00} - 1)c^2/2$.

The quest for direct detection of GR's cardinal predictions - black holes (BHs) and gravitational waves (GWs), is still on. Sections 2 and 3 briefly discuss elements of BH physics. In section 4, I describe some ongoing work of ours on GWs from rapidly rotating, magnetized white dwarfs (WDs) .

2. Cosmic Phenomena : General Relativity to the rescue

Multitude of astrophysical and cosmological observations find natural explanations after GR is taken into consideration. For instance, the observed high redshifts ($1 \lesssim z \lesssim 7$) of many extragalactic sources like quasars and radio-galaxies are unlikely to be Doppler shifts, as that would imply galactic size objects undergoing relativistic motion with high Lorentz factor $\gamma \equiv (1 - v^2/c^2)^{-1/2}$ entailing not only their gaseous contents being stripped off as they plough through the intergalactic medium with speed

$v \approx c$, but also having greater observed mass, $\gamma\, M_{gal}$, where M_{gal} is the mass in their rest frames. Since, neither of these have any observational support, the explanation of redshift due to cosmological stretching of wavelengths as the universe expands has stood the test of time. Solutions representing expanding (or contracting) universe for homogeneous and isotropic matter distribution ensue from GR itself.

GR even allows repulsive gravity provided there exists an exotic matter whose pressure is sufficiently negative. If one has a spherically symmetric object with uniform density ρ and pressure p, it can be shown by going beyond the Newtonian approximation that the acceleration of a test particle near the surface of the object is given by,

$$\frac{d^2\vec{r}}{dt^2} \cong -\frac{4}{3}\pi G\left[\rho + \frac{3p}{c^2}\right]\vec{r}\,. \tag{3}$$

For an object made up of some kind of dark energy (DE) with $p < -\rho c^2/3$, eq.(3) entails repulsive gravity. Cosmological constant Λ is a special case of DE with $p = -\rho c^2$. This line of reasoning indeed explains the observed late time acceleration of the universe deduced from SN Ia data as well as precision studies of cosmic microwave background radiation.

Very large luminosities, jets and rapid time variability associated with active galactic nuclei (AGNs) can very effectively be explained in terms of accretion of matter close to the event horizon of a supermassive black hole (SMBH) using the formalism of curved space-time magnetohydrodynamics and plasma physics. To illustrate in a simplified manner why BH accretion is an efficient generator of energy, one may consider a tiny volume element dV of an accretion disc, having rest mass dm, going around a Schwarzschild BH (SchBH) in a quasi-Keplerian orbit. Since $\xi^\mu = \delta^\mu_0$ is a time-like Killing vector in the vicinity of a SchBH, the volume element orbiting at a radial coordinate r with angular speed $d\phi/dt$ has energy,

$$dE = \frac{c^2(1 - R_s/r)\, dm}{\sqrt{1 - R_s/r - (r/c)^2(d\phi/dt)^2}} \tag{4}$$

where $R_s \equiv 2GM/c^2$ is the Schwarzschild radius of the BH. As the innermost stable circular orbit (ISCO) around a SchBH has a radius, $r_{ISCO} = 3R_s$ corresponding to an angular speed $|d\phi/dt| = c/(3\sqrt{6}R_s)$, one has $dE = 2\sqrt{2}\; c^2\; dm/3$ from eq.(4), implying a radiative loss of energy $c^2\; dm - dE \cong -0.06c^2\; dm$, as the matter spiraled in from very large r to r_{ISCO}. Hence, accretion on to a SchBH can transform rest energy of matter into radiation with a maximum of $\sim$ 6% efficiency, provided the viscous dissipation of kinetic energy due to gradient of speed in the disc is efficient enough. Studies indicate that large magnetic fields threading through accretion discs help in making ISCO come closer to the event horizon, leading to increase in the efficiency of AGNs (Piotrovich et al. 2014).

Radiation from blazars and other AGNs display rapid fluctuations. High frequency radiation from blazars vary on time scales $\Delta t \sim$ few hours (McHardy et al.

2006). Causality arguments imply that transient processes taking place close to the event horizon of the SMBH can give rise to $\Delta t \sim \kappa \, r_{ISCO}/c$, where κ is a dimensionless parameter $\gtrsim 1$. Thus, observed $\Delta t \sim$1 hour time scale in a blazar would mean $R_s \sim 3.6 \times 10^{14}/\kappa$ cm, indicating a SMBH of mass $\sim 10^8 M_{\odot}$. As most astrophysical objects exhibit rotation, a BH formed out of the collapse of a stellar system is very likely a Kerr BH. In addition to having an event horizon, a Kerr BH is also endowed with an ergosphere, a region where test particles cannot have constant spatial Boyer-Lindquist coordinates. Counter-rotating test particles in the ergosphere can have negative energy, as the conserved energy $E = p^{\mu}\xi_{\mu}$ can be negative in this region ($\xi^{\mu} = \delta^{\mu}_{0}$ is a Killing vector). Penrose (1969) had proposed a mechanism to extract rotational energy of a Kerr BH, wherein a particle of energy E_1 decays into two particles with energy E_2 and E_3 after it enters the ergosphere. Since $E_1 = E_2 + E_3$, the second particle escapes with energy $E_2 > E_1$ if $E_3 < 0$ (Penrose 1969, Bhat et al. 1985). Detection of Penrose process related physics in AGNs and galactic X-ray sources or of GWs due to quasi-normal modes during BH formation/perturbation are likely to bolster evidence for BH existence.

3. Black holes and Quantum Theory

Quantum vacuum is spontaneously and randomly creating virtual pairs of particles of mass m that last only for short time $\sim \hbar/(2mc^2)$, consistent with uncertainty principle. This is prevalent near a BH. According to eq.(4), it costs very little energy to create virtual pairs at rest (i.e. $d\phi/dt = 0$) near $r \approx R_s$, so that if one of the pair falls into the BH, the other can escape to infinity. Hawking (1975) had shown that particles so produced, from the vicinity of a SchBH of mass M, have a thermal spectrum with temperature,

$$T_H = \frac{\hbar c^3}{8\pi G k_B M} = \frac{m_{Pl}^2 c^2}{8\pi k_B M} \cong 10^{26}\ {}^{\circ}\mathrm{K} \left(\frac{M}{1\ \mathrm{gm}}\right)^{-1} \tag{5}$$

where $m_{Pl} = \sqrt{\hbar c/G}$ is the Planck mass. Eq.(5) succinctly captures elements of gravity, quantum theory and statistical mechanics.

According to quantum theory, physical states evolve unitarily via Schrödinger equation so that a pure state $|\psi>$ evolves into another pure state $|\chi>$. A BH can form by gravitational collapse of a system described by a pure state (e.g. a macroscopic Bose-Einstein condensate at zero temperature (Das Gupta 2015)). BH formation and evolution too should be governed by unitary time evolution. In which case, final state of the BH along with its entourage of Hawking radiation must correspond to a pure state. On the contrary, Hawking radiation is thermal in nature, corresponding to a mixed state, and hence cannot arise through unitary evolution of a pure state. A possible resolution is that the BH state is quantum mechanically entangled with the state of the radiated particles, so that the state of the combined system is pure. But then, what happens when the BH evaporates completely? This, so called BH information loss problem, is causing novel ideas to emerge, ranging from fuzz ball to fire wall (Hawking 2005, Chen, Ong and Yeom 2015).

It is interesting to deduce, using simple arguments, that Hawking radiation prevents formation of bound quantum state comprising of a BH and an elementary particle. The Bohr model describing a BH of mass M and a test particle of mass m bound gravitationally, predicts quantized states with energy (however, for accurate energy eigenvalues, see (Lasenby et al. 2005)),

$$E_n = -\frac{G^2 M^2 m^3}{2\, n^2 \hbar^2} = -\frac{G\, M\, m}{2\, r_n} \tag{6a}$$

corresponding to orbital radii,

$$r_n = \frac{n^2 \hbar^2}{G\, M\, m^2} \quad \text{with} \quad n = 1, 2, ... \tag{6b}$$

For a stable bound state, it is necessary that the closest orbital radius satisfies $r_1 > R_s$, otherwise the particle will simply be swallowed by the BH. Hence,

$$M < \frac{m_{Pl}^2}{\sqrt{2}\, m} \tag{7a}$$

so that from eq.(5) the corresponding Hawking temperature is,

$$T_H > \frac{\sqrt{2}\, mc^2}{8\pi k_B}\ . \tag{7b}$$

From eqs.(6a) and (6b), the binding energy E_B of such a system is,

$$E_B = \frac{G\, M\, m}{2\, r_1} < \frac{G\, M\, m}{2\, R_s} < \frac{1}{4} mc^2\ . \tag{7c}$$

While, from eqs.(7b) and (7c), typical thermal energy of a particle emitted due to Hawking process is $\sim k_B T_H > \sqrt{2}\, mc^2/8\pi$, which is $\gtrsim E_B$. Therefore, Hawking radiation does break up such a 'black hole atom'.

4. Gravitational Radiation from White Dwarfs

Magnetic fields B in white dwarfs (WDs) vary over a wide range. Some WDs have B as high as 10^9 G on their surface (Schmidt et al. 2003, Liebert et al. 2005, Kawka et al. 2007). Magnetic stress induced by large interior B can make a spinning WD non-axisymmetric, leading to emission of GWs (Heyl 2000). The WD in AE Aquarii has a spin period of only 33 s and a spin down rate of $\dot{P} = (5.64 \pm 0.02) \times 10^{-14}\ s\ s^{-1}$, implying a magnetic dipole moment $\mu \approx 1.5 \times 10^{34}$ G cm^3. Its polar B is likely to be about 10^8 G (Ikhsanov and Beskrovnaya 2012). As of now, the fastest spinning WD is associated with the binary system RX J0648.0-4418 having spin period of 13.18 s and mass $1.28 \pm 0.05\ M_\odot$ (Mereghetti et al. 2009, Mereghetti 2013, Mereghetti et al. 2013).

The luminosity due to magnetic dipole radiation from a compact magnetized source that is spinning with an angular speed Ω and located at a distance d from us is given by $L_{EM} = (2\,\mu^2\ \sin^2\alpha\ \Omega^4)/3\,c^3$, where $\mu = \frac{1}{2}B_pR^3$ is the magnetic dipole moment of the object and α is the angle between the spin axis and the magnetic dipole. While the GW amplitudes from the rotating WD are given by (Heyl 2000), $h_\oplus = h_0\sin^2\alpha\ \cos[2\Omega(t-t_0)]$ and $h_\otimes = h_0\sin^2\alpha\ \sin[2\Omega(t-t_0)]$ where,

$$h_0 \equiv -\frac{6G\,I_{zz}\Omega^2}{c^4\,d}$$

and the reduced mass quadrupole moment (assuming the z-axis to be along the magnetic dipole moment),

$$I_{zz} \equiv \int \rho(z^2 - \frac{1}{3}r^2)d^3r = -\beta_6\ \delta_M\ M\,R^2$$

with parameter δ_M being defined as the ratio of magnetic energy to gravitational potential energy,

$$\delta_M = \frac{\int (B^2/8\pi)d^3r}{(\alpha_3 GM^2/R)} \approx \frac{R^4 < B^2 >}{6\alpha_3 GM^2}\ .$$

The parameters α_3 and β_6 are of order unity whose values depend on the mass density profile. The GW energy flux is given by,

$$F_{GW} = \frac{c^3}{16\pi G}[\dot{h}_\oplus^2 + \dot{h}_\otimes^2] = \frac{c^3}{4\pi G}h_0^2\ \Omega^2\ \sin^4\alpha$$

from which one can estimate the GW luminosity to be,

$$L_{GW} \approx 4\pi d^2 F_{GW} = \left(\frac{\beta_6}{\alpha_3}\right)^2 \frac{< B^2 >^2\ R^{12}\ \Omega^6 \sin^4\alpha}{GM^2c^5}$$

Therefore, the ratio of GW to EM power is given by,

$$\frac{L_{GW}}{L_{EM}} = 6\left(\frac{\beta_6}{\alpha_3}\right)^2 \frac{< B^2 >^2}{B_p^2}\quad \frac{R^6\ \Omega^2 \sin^2\alpha}{GM^2c^2} \tag{8a}$$

$$= 37.5\left(\frac{\beta_6}{\beta_I\alpha_3}\right)^2 \frac{< B^2 >^2}{B_p^2}\quad \frac{R^2\ J^2 \sin^2\alpha}{GM^4c^2} \tag{8b}$$

where $J = 0.4\beta_I\ M\ R^2\Omega$ is the WD's spin angular momentum with β_I being a parameter of order unity whose value depends on the mass distribution.

For AE Aquarii, a cataclysmic variable (Ikhsanov and Beskrovnaya 2012): $R = 10^{8.8}$ cm, $B_p \approx 10^8$ G, $P = 33$ s, $M = 0.65\ M_\odot - 1.2\ M_\odot$, $\alpha = 76^\circ - 78^\circ$ so that from eq.(8a),

$$\frac{L_{GW}}{L_{EM}} = 0.54\left(\frac{\beta_6}{\alpha_3}\right)^2\left(\frac{R}{10^{8.8}\ cm}\right)^6\left(\frac{P}{33\ s}\right)^{-2}\left(\frac{M}{1\ M_\odot}\right)^{-2}\left(\frac{B_p}{10^8\ G}\right)^{-2}\left(\frac{< B^2 >}{(10^{11}\ G)^2}\right)^2\left(\frac{\sin\alpha}{\sin 77^\circ}\right)^2$$

While for the WD in RX J0648.0-4418 system (Mereghetti et al. 2009, Mereghetti et al. 2012, Mereghetti et al. 2013): $R = 3\times10^8$ cm, $P = 13.184$ s, $M = 1.28\ M_\odot$. If one assumes the other parameters to have same values as in the former case, one obtains $L_{GW}/L_{EM} \sim 0.02$ for RX J0648.0-4418.

Assuming the WD to be in quasi-hydrostatic equilibrium, one may employ the scalar virial theorem (SVT) (Shapiro and Teukolsky 1983),

$$2T + W + 3\Pi + \mathcal{M} = 0\ , \tag{9}$$

where $T \equiv (\kappa_3\ J^2)/(2M\ R^2)$, $W \equiv -(\alpha_3\ G\ M^2)/R$, $3\Pi \equiv (\beta_3\ M^{4/3})/R$ and $\mathcal{M} \equiv \int (B^2/8\pi) d^3r$ are the energies associated with rotation, gravitation, degeneracy pressure and the magnetic field, respectively. The total energy of the WD is,

$$E = T + W + 3\Pi + \mathcal{M}\ . \tag{10}$$

Eqs.(9) and (10) imply $E = -T$ so that loss of energy incurred because of emission of GWs and EM waves leads to,

$$\frac{dE}{dt} = -\frac{dT}{dt} = -(L_{EM} + L_{GW}) \tag{11}$$

Hence, from eq.(11), one finds $\frac{dT}{dt} > 0$, indicating that as the rapidly spinning WD loses energy by radiating GWs as well as EM waves, the rotational kinetic energy of the compact object tends to increase with time. Depending on whether the loss of energy due to radiation causes the WD to grow or shrink in size will determine whether it slows down or spins up (also, see Shapiro, Teukolsky and Nakamura 1990, Boshkayev et al. 2013). Furthermore, in the framework of SVT, the rate of change of the WD's period of rotation as it emits GWs and EM waves are given by,

$$\left|\frac{dP}{dt}\right|_{GW} = 2.7\times10^3\left(\frac{\alpha_3\ T}{0.16\kappa_3\beta_I^2\ |W|}\right)^{2/3}\left(\frac{\beta_6\delta_M \sin^2\alpha}{\sqrt{\beta_I}}\right)^2\left(\frac{G\ M\ \Omega}{c^3}\right)^{5/3}$$

and,

$$\left|\frac{dP}{dt}\right|_{EM} = \frac{5\pi^2}{P}\frac{B_p^2\ R^4\ \sin^2\alpha}{\beta_I\ M\ c^3}\ ,$$

respectively.

Conclusions

Since strong gravity conditions prevail near compact objects, it is likely that imprints of new GR effects will show up in AGN, GRB and neutron star/magnetar studies involving data from sensitive and high resolution telescopes like ASTROSAT, TMT and SKA. Energy loss due to GWs could cause spin up or spin down of highly magnetized and rapidly rotating WDs. Such observations would indicate the extent of WD deformation induced by large magnetic fields. Detection of GWs from rapidly spinning WDs using space borne instruments like LISA could shed light on their interior magnetic fields.

Acknowledgements

It is a pleasure to thank Indranil Chattopadhyay and his co-organizers of RETCO-II held at ARIES, Nainital, for creating an academically stimulating ambience during the conference. I also thank the anonymous referee for useful suggestions.

References

Bhat M., Dhurandhar S., Dadhich N., 1985, J. Astrophys. Astr. 6, 85

Boshkayev K., Rueda J. A., Ruffini R., Siutsou I., 2013, ApJ, 762, 117

Chen P., Ong Y. C., Yeom D., 2015, arXiv:1412.8366v2 [gr-qc], and references therein.

Das Gupta P., 2015, arXiv:1505.00541v1 [gr-qc] (to appear in Current Science)

Hawking S. W., 1975, Commun. Math. Phys. 43, 199

Hawking S. W., 2005, PRD, 72, 084013

Heyl J. S., 2000, MNRAS, 317, 310

Ikhsanov N.R., Beskrovnaya N.G., 2012, Astronomy Reports, 56, 595

Kawka A., Vennes S., Schmidt G. D., Wickramasinghe D. T., Koch, R., 2007, ApJ, 654, 499

Lasenby A. N., Dolan C. J. L., Pritchard J., Caceres A., Dolan S. R., 2005, PRD, 72, 105014

Liebert J., et al., 2005, Astron. J., 129, 2376

McHardy I. M, Koerding E., Knigge C., Uttley P., Fender R. P., 2006, Nature, 444, 730

Mereghetti S., Tiengo A., Esposito P., La Palombara N., Israel G. L., Stella L., 2009, Science, 325, 1222

Mereghetti S., 2013, Proceedings of 13th Marcel Grossman Meeting (MG13), Stockholm, Sweden (2012), arXiv:1302.4634

Mereghetti S., La Palombara N., Tiengo A., Sartore N., Esposito P., Israel G. L., Stella, L., 2013, arXiv:1304.1653

Penrose R., 1969, Rivista del Nuovo Cimento

Piotrovich M. Yu., Gnedin Yu. N., Silant'ev N. A., Natsvlishvili T.M., Buliga, S. D., 2014, arXiv:1410.1663v2, and references therein.

Schmidt G. D., et al., 2003, ApJ, 595, 1101

Shapiro S. L., Teukolsky S. A., 1983, Black Holes, White Dwarfs and Neutron Stars, John Wiley and Sons, Inc.

Shapiro S. L., Teukolsky S. A., Nakamura T., 1990, ApJL, 357, L17